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The Lawrence–Krammer representation is a quantization of the symmetric square of the Burau representation ☆

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We show that the Lawrence–Krammer representation is a quantization of the symmetric square of the Burau representation. Here, by quantization we mean changing the natural numbers by q-natural numbers and the corresponding quantization of the Pascal triangle, appearing naturally in representations of the braid groups. This connection allows us to construct new representations of the braid groups.

☆ This work is dedicated to Ukraine and to all fearless Ukrainians defending their country.

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Contents

1. Introduction

The *braid group* B_n was introduced by Artin [3] in 1926 to *study knots*:

$$B_n = \langle (\sigma_i)_{i=1}^{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \le i \le n-2,\ \sigma_i\sigma_j = \sigma_i\sigma_j,\ \mid i-j \mid \ge 2\rangle. \tag{1.1}$$

Despite many very profound results in the *classifications of knots*, the general problem remains unsolved. We mention only some polynomial invariants without any details: Alexander polynomials, Jones polynomials, Kauffman polynomials, Reshetikhin-Turaev quantum polynomials, Vasiliev invariants etc. The same is true for the *classification of the representation* of B_n. See e.g. [30] for representations of B_3.

Our main aim is to connect two well-known representations of the braid groups: the Lawrence–Krammer and the Burau representations of B_n.

In 1936 W. Burau introduced [10] the family of representations of the braid group B_n as the *deformation* of the *standard representation* of the symmetric group S_n, see (2.1). The Burau representation for $n = 2, 3$ has been known to be *faithful* for some time, i.e., *the kernel of the corresponding representation is trivial.* Indeed, in 1969 Magnus and Peluso [23] showed that the Burau and Gasner representations are faithful for $n = 2, 3$. See also J. Birman [8, Theorem 3.15], [9]. Moody [24] in 1991 showed that the Burau representation is not faithful for $n \ge 9$; this result was improved to $n \ge 6$ by Long and Paton in 1992, [22]. The non-faithfulness for $n = 5$ was shown by Bigelow in 1999, [4]. The faithfulness of the Burau representation for $n = 4$ is an *open problem.*

The Burau representation appears as a summand of the *Jones representation* [13], and for $n = 4$, the faithfulness of the Burau representation is equivalent to that of the Jones representation, which furthermore is related to the question of whether or not the Jones polynomial is an *unknot detector* [7]. The problem of the *linearity*, i.e., the existence of the faithful representation, for the braid groups B_n for $n \geq 4$ remains open until 2000. In 1990 R. Lawrence [20] constructed a two-parameter family of representation of the braid group B_n, using homological methods. In 2000, D. Krammer [17], using completely algebraic methods constructed the same representations and showed that this representation is faithful for B_4. One year later, in 2001, S. Bigelow showed that the *Lawrence–Krammer* (LK) representation is faithful for all B_n. Next year, D. Krammer [18] proved the same result by a different method. Finally, it was established that the braid groups B_n are linear for all n: in particular, for $n \leq 3$, one can use Burau and for $n \geq 4$ one can use LK representations.

We show that the *Lawrence–Krammer representation is a quantization of the symmetric square of the Burau representation*. The hint is the following: both representations of B_n have the same dimensions $\frac{n(n-1)}{2}$ and their spectrums coincide, see Remark 2.4. This connection allows us to construct new representations of the braid groups.

Another three hints are as follows: Lickorish [21] observed that the Kauffman polynomial (related to the BMW algebras) at a particular parameter choice $r = q^3$ is the same as the square of the Jones polynomial at some parameter choice. This paper investigates the relationships and inter-connections between various polynomial invariants of classical links, particularly the Jones polynomial.

Zinno [32] showed that the LK representation comes from the corresponding Birman–Murakami–Wenzl (BMW) algebra, i.e. for $r = q^3$ (his parametrization is a little different). A connection is made between the LK representation and the BMW algebra. Inspired by a dimension argument, a basis is found for a certain ireducible representation of the algebra, and relations which generate the matrices are found.

Finally Larsen and Rowell showed [19] that the symmetric square of the Temperley-Lieb algebra was related to the same BMW algebra. More precisely, they establish isomorphisms between certain specializations of Birman-Murakami-Wenzl algebras and the symmetric squares of Temperley-Lieb (TL) algebras. Thus the fact that the reduced Burau is a representation of the TL-algebras, and the LK is a representation of certain BMW-algebras suggests that the main result of our paper is a manifestation of this.

Formanek et al. [12] also has some classification results for all B_n. Namely, they classified all representations for B_n in dimension $\leq n$.

The *content* of the article is as follows. In Section 2.1 we recall the definition of the Burau representation for B_n, in Section 2.3 we define the Lawrence–Krammer representation $K_n^{(t,q)}$ with the corrections made by Bigelow. In Lemma 2.1 we establish the connection between $k_n^{(t,q)}$ and $k_{n+1}^{(t,q)}$, see (2.17). Some generalizations of Burau representation are done in Section 2.4. The main result, Theorem 2.3, is stated in Section 2.5. The *Pascal triangle* and the connection with representations of the group B_3 and with the representations of a Lie algebra $\mathfrak{sl}_2$ is done in Section 2.8. The important object *q-Pascal* *triangle* and representations of B_3, see [16], connected to the Tuba-Wenzl results [28,29], are explained in Section 2.9–2.10. In Section 2.11 we define two symmetric bases of $V \otimes V$. The proof of Theorem 2.3 is done in Section 2.12. The essential ingredient is Lemma 2.1. Section 3.1 is devoted to the possible way of constructing new representations of B_n, starting from representations of the Lie algebra $\mathfrak{sl}_{n-1}$.

2. Connection between the Lawrence–Krammer and the Burau representation

2.1. The Burau representation of B_n

The Burau representation $\rho : B_n \to \mathrm{GL}_n\big(\mathbb{Z}[t,t^{-1}]\big)$ of B_n is defined for a non-zero complex number t by

$$\sigma_i \mapsto I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right) \oplus I_{n-i-1}, \tag{2.1}$$

where $1-t$ is the (i,i) entry. For $t = 1$ this is the *standard representation* of the symmetric group S_n interchanging by matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ two neighbors basic elements e_i and e_{i+1} in the space $\mathbb{C}^n$. The vector $e = e_1 + \cdots + e_n$ is invariant for ρ, therefore the representation ρ splits into 1-dimensional and an $n-1$-dimensional irreducible representations, known as the *reduced Burau representation* $\rho_n^{(t)} : B_n \to \mathrm{GL}_{n-1}\big(\mathbb{Z}[t,t^{-1}]\big)$

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix} \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto I_{n-3} \oplus \begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix}, \tag{2.2}$$

$$\sigma_i \mapsto I_{i-2} \oplus \begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus I_{n-i-2},\ 2 \le i \le n-2. \tag{2.3}$$

We use the following equivalent form of the reduced Burau representation $\rho_n^{(t)} : B_n \mapsto \mathrm{GL}_{n-1}\big(\mathbb{Z}[t,t^{-1}]\big)$, compare with (2.2) and (2.3):

$$\sigma_1 \mapsto \left(\begin{smallmatrix} -t & t \\ 0 & 1 \end{smallmatrix}\right) \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ 1 & -t \end{pmatrix}, \tag{2.4}$$

$$\sigma_i \mapsto I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2},\ 2 \le i \le n-2. \tag{2.5}$$

The equivalence is given by

$$j^{-1}\begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix} j = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -t & 0 \\ 0 & t & 1 \end{pmatrix} \overset{(\cdot)^T}{\mapsto} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad j = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and A^T is the matrix transposed to A.

2.2. The reduced Burau representation of B_∞

Define the group B_∞ as follows

$$B_\infty=\langle(\sigma_i)_{i\in\mathbb{Z}}\ |\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\ i\in\mathbb{Z},\ \sigma_i\sigma_j=\sigma_i\sigma_j,\ |\ i-j\ |\geq 2\rangle. \tag{2.6}$$

Consider a complex Hilbert space

$$H=\ell_2(\mathbb{Z})=\{x=(x_k)_{k\in\mathbb{Z}}\mid \|x\|^2=\sum_{k\in\mathbb{Z}}|x_k|^2<\infty\} \tag{2.7}$$

with its standard orthonormal basis $e=(e_k)_{k\in\mathbb{Z}}$ defined by $e_k=(\delta_{kn})_{n\in\mathbb{Z}}$. Define *the reduced Burau representation of the group* B_∞ in the space H as follows:

$$\sigma_i\mapsto b_{i,\infty}=I_\infty\oplus\begin{pmatrix}1&0&0\\1&-t&t\\0&0&1\end{pmatrix}\oplus I_\infty,\quad i\in\mathbb{Z}, \tag{2.8}$$

where $-t$ is the (i,i) entry. It is clear that $b_{i+1,\infty}$ is obtained from $b_{i,\infty}$ by shifting the basis $e: e_n\to e_{n+1}$. More precisely, if we denote by J the unitary operator on $\ell_2(\mathbb{Z})$ defined as follows: $Je_n=e_{n+1},\ n\in\mathbb{Z}$ we get representation

$$Jb_{i,\infty}J^{-1}=b_{i+1,\infty}. \tag{2.9}$$

Next, denote by P_n the orthogonal projector of H on the subspace H_n *generated by the first n vectors*, i.e., $H_n=\langle e_k\mid 1\leq k\leq n\rangle$. Then we get the reduced Burau representation: of B_{n+1}:

$$P_nb_{k,\infty}P_n=\rho_{n+1}^{(t)}(\sigma_k),\quad 1\leq k\leq n. \tag{2.10}$$

Denote by $J(n)$ the unitary operator on H_n defined as follows

$$J(n)e_k=e_{k+1},\quad 1\leq k\leq n-1,\quad J(n)e_n=e_1, \tag{2.11}$$

and by i_n the natural embedding of $\mathrm{GL}(n-1,\mathbb{C})$ into $\mathrm{GL}(n,\mathbb{C})$ defined as follows:

$$\mathrm{GL}(n-1,\mathbb{C})\ni x\to i_n(x)=x+E_{nn}\in\mathrm{GL}(n,\mathbb{C}). \tag{2.12}$$

Then we get for $n\geq 3,\ 1\leq k\leq n-2$ and $n-1\leq r\leq n$

$$\rho_{n+1}^{(t)}(\sigma_k)=i_n\big(\rho_n^{(t)}(\sigma_k)\big)\ \text{ and }\ \rho_{n+1}^{(t)}(\sigma_r)=J(n)i_n\big(\rho_n^{(t)}(\sigma_{r-1})\big)J(n)^{-1}. \tag{2.13}$$

Remark 2.1. Formula (2.13) allows us to construct the reduced Burau representation for B_{n+1} from the reduced Burau representation for B_n. For $n=4$ we have

$$\rho_4^{(t)}(\sigma_1)=\begin{pmatrix}-t&t&0\\0&1&0\\0&0&1\end{pmatrix},\ \rho_4^{(t)}(\sigma_2)=\begin{pmatrix}1&0&0\\1&-t&t\\0&0&1\end{pmatrix},\ \rho_4^{(t)}(\sigma_3)=\begin{pmatrix}1&0&0\\0&1&0\\0&1&-t\end{pmatrix},$$

$$\rho_5^{(t)}(\sigma_1)=\begin{pmatrix}-t&t&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \rho_5^{(t)}(\sigma_2)=\begin{pmatrix}1&0&0&0\\1&-t&t&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \rho_5^{(t)}(\sigma_3)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&-t&t\\0&0&0&1\end{pmatrix},$$

$$\rho_5^{(t)}(\sigma_4)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&-t\end{pmatrix},\ \text{so we get for } 1\le k\le 2, \text{ and for } 3\le r\le 4$$

$$\rho_5^{(t)}(\sigma_k)=i_n\big(\rho_4^{(t)}(\sigma_k)\big)\ \text{ and }\ \rho_5^{(t)}(\sigma_r)=J(4)i_4\big(\rho_4^{(t)}(\sigma_{r-1})\big)J(4)^{-1}. \tag{2.14}$$

2.3. The Lawrence-Krammer representation of the braid group B_n

The *Lawrence-Krammer representation* $K_n^{(t,q)}$ of the Braid group B_n was introduced by Lawrence in [20] (1990), and studied by Krammer in [17,18] and Bigelow in [5,6]. To give formulas, we follow Bigelow [5] and the corrected version in [6]. The representation $K_n^{(t,q)}$ is the following action of B_n on a free module V of rank $\binom{n}{2}=\frac{n!}{2!(n-2)!}$ with basis $\{F_{i,j} : 1\le i<j\le n\}$:

$$\sigma_i(F_{j,k})=\begin{cases}F_{j,k} & i\notin\{j-1,j,k-1,k\},\\ qF_{i,k}+q(q-1)F_{i,j}+(1-q)F_{j,k} & i=j-1,\\ F_{j+1,k} & i=j\neq k-1,\\ qF_{j,i}+(1-q)F_{j,k}+q(1-q)tF_{i,k} & i=k-1\neq j,\\ F_{j,k+1} & i=k,\\ -tq^2F_{j,k} & i=j=k-1.\end{cases} \tag{2.15}$$

From [6]: "This action is given in [17], except that Krammer's $-t$ is my t. It is also in [5], but with a sign error. The name "Krammer representation" was chosen because Krammer seems to have initially found this independently of Lawrence and without any use of homology". We would like to change notations and define the *Lawrence-Krammer representation* $k_n^{(t,q)}$ as follows:

$$\sigma_i(F_{j,k})=\begin{cases}F_{j,k} & i\notin\{j-1,j,k-1,k\},\\ tF_{i,k}+(1-t)F_{j,k}+t(t-1)F_{i,j} & i=j-1,\\ F_{j+1,k} & i=j\neq k-1,\\ tF_{j,i}+(1-t)F_{j,k}+qt(t-1)F_{i,k} & i=k-1\neq j,\\ F_{j,k+1} & i=k,\\ qt^2F_{j,k} & i=j=k-1.\end{cases} \tag{2.16}$$

Connection between two representations is as follows:

$$K_n^{(q,-t)}(\sigma_r)=k_n^{(t,q)}(\sigma_r),\quad 1\le r\le n-1. \tag{2.17}$$

We think that this choice of the parameters reflects better the situation since, historically Burau used parameter t, and q, in our notations, is really *a parameter of quantization*, see (2.58). In particular, the LK representations $k_n^{(t,q)}$ in notations (2.16) for B_n with $2 \leq n \leq 6$ are as follows[1]:

$$k_2^{(t,q)}(\sigma_1) = qt^2, \tag{2.18}$$

$$k_3^{(t,q)}(\sigma_1) = \left(\begin{array}{c|cc} qt^2 & 0 & t(t-1) \\ \hline 0 & 0 & t \\ 0 & 1 & 1-t \end{array}\right), \quad k_3^{(t,q)}(\sigma_2) = \left(\begin{array}{cc|c} 0 & t & 0 \\ 1 & 1-t & 0 \\ \hline 0 & qt(t-1) & qt^2 \end{array}\right), \tag{2.19}$$

$$k_4^{(t,q)}(\sigma_1) = \begin{pmatrix} qt^2 & 0 & 0 & t(t-1) & t(t-1) & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 1-t & 0 & 0 \\ 0 & 0 & 1 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad k_4^{(t,q)}(\sigma_2) = \left(\begin{array}{ccc|ccc} 0 & t & 0 & 0 & 0 & 0 \\ 1 & 1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & qt(t-1) & 0 & qt^2 & 0 & t(t-1) \\ 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & 1-t \end{array}\right),$$

$$k_4^{(t,q)}(\sigma_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 1-t & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 1-t & 0 \\ 0 & 0 & qt(t-1) & 0 & qt(t-1) & qt^2 \end{array}\right). \tag{2.20}$$

$$k_5^{(t,q)}(\sigma_1) = \begin{pmatrix} qt^2 & 0 & 0 & 0 & t(t-1) & t(t-1) & t(t-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1-t & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1-t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$k_5^{(t,q)}(\sigma_2) = \left(\begin{array}{cccc|cccccc} 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & qt(t-1) & 0 & 0 & qt^2 & 0 & 0 & t(t-1) & t(t-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1-t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$k_5^{(t,q)}(\sigma_3) = \left(\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & qt(t-1) & 0 & 0 & qt(t-1) & 0 & qt^2 & 0 & t(t-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1-t \end{array}\right),$$

[1] Together with Prof. Yang Hui-He and Dr. Mikhail Burtsev, both from LIMS, we found the formulas for B_5.

$$k_5^{(t,q)}(\sigma_4)=\left(\begin{array}{cccc|cccccc}
1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&t&0&0&0&0&0&0\\
0&0&1&1-t&0&0&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&t&0&0&0\\
0&0&0&0&0&1&1-t&0&0&0\\
0&0&0&0&0&0&0&0&t&0\\
0&0&0&0&0&0&0&1&1-t&0\\
0&0&0&qt(t-1)&0&0&qt(t-1)&0&qt(t-1)&qt^2
\end{array}\right). \tag{2.21}$$

The formulas of LK representation $k_n^{(t,q)}(\sigma_r)$ for B_6 are as follows:

$$k_6^{(t,q)}(\sigma_1)=\begin{pmatrix}
qt^2&0&0&0&0&t(t-1)&t(t-1)&t(t-1)&t(t-1)&0&0&0&0&0&0\\
0&0&0&0&0&t&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&t&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&t&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&t&0&0&0&0&0&0\\
0&1&0&0&0&1-t&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1-t&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1-t&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1-t&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}, \tag{2.22}$$

$$k_6^{(t,q)}(\sigma_2)=\left(\begin{array}{ccccc|cccccccccc}
0&t&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1-t&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
\hline
0&qt(t-1)&0&0&0&qt^2&0&0&0&t(t-1)&t(t-1)&t(t-1)&0&0&0\\
0&0&0&0&0&0&0&0&0&t&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&t&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&t&0&0&0\\
0&0&0&0&0&0&1&0&0&1-t&0&0&t&0&0\\
0&0&0&0&0&0&0&1&0&0&1-t&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1-t&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right),$$

$$k_6^{(t,q)}(\sigma_3)=\left(\begin{array}{ccccc|cccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&t&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1-t&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&t&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1-t&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&qt(t-1)&0&0&0&qt(t-1)&0&0&qt^2&0&0&t(t-1)&t(t-1)&0\\
0&0&0&0&0&0&0&0&0&0&0&0&t&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&t&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1-t&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1-t&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right),$$

$$k_6^{(t,q)}(\sigma_4)=\left(\begin{array}{ccccc|cccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&t&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1-t&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&t&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1-t&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&t&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1-t&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&qt(t-1)&0&0&0&qt(t-1)&0&0&qt(t-1)&0&qt^2&0&t(t-1)\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&t\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1-t
\end{array}\right),$$

$$k_6^{(t,q)}(\sigma_5)=\left(\begin{array}{ccccc|cccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&t&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1-t&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&t&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1-t&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&t&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1-t&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&t&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1-t&0\\
0&0&0&0&qt(t-1)&0&0&0&qt(t-1)&0&0&qt(t-1)&0&qt(t-1)&qt^2
\end{array}\right).$$

To connect representations $k_{n+1}^{(t,q)}$ and $k_n^{(t,q)}$ we introduce some notations. Define for $\sigma \in S_n$, where S_n is a permutation group, and

$$\sigma=\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n-1) & \sigma(n)\end{pmatrix},$$

the *basis permutation operator* $s_\sigma \in \mathrm{GL}(n,\mathbb{C})$, defined by

$$s_\sigma e_k = e_{\sigma(k)}, \quad 1 \le k \le n, \tag{2.23}$$

where $(e_k)_{k=1}^n$ is a standard basis in $\mathbb{C}^n$.

Lemma 2.1. [2] *We have the following relations between $k_n^{(t,q)}$ and $k_{n+1}^{(t,q)}$:*

$$k_{n+1}^{(t,q)}(\sigma_r)-qt(t-1)E_{(1,r+1),(r,r+1)}=D_{n+1,r}\oplus k_n^{(t,q)}(\sigma_{r-1}),\ 2\le r\le n, \tag{2.24}$$
$$s_{\sigma_{n+1,1}}\left(k_{n+1}^{(t,q)}(\sigma_1)-t(t-1)E_{(2,n+1),(1,2)}\right)s_{\sigma_{n+1,1}}^{-1}=k_n^{(t,q)}(\sigma_1)\oplus D_{n+1,1},$$

where $D_{n+1,r}$ and $s_{\sigma_{n+1,1}}$ are defined below.

[2] After proving this lemma, the author asked Dr. Mikhail Burtsev from LIMS to find the connection between $k_n^{(t,q)}$ and $k_{n+1}^{(t,q)}$ with GPT o3. The result was not the same but rather similar.

Proof. If we compare (2.19) and (2.18) we get

$$k_3^{(t,q)}(\sigma_2)-qt(t-1)E_{13,23}=D_{32}\oplus k_2^{(t,q)}(\sigma_1), \tag{2.25}$$

$$k_3^{(t,q)}(\sigma_1)-t(t-1)E_{23,12}=k_2^{(t,q)}(\sigma_1)\oplus D_{31}, \tag{2.26}$$

$$\text{where } D_{32}=D_{31}=\begin{pmatrix}0&t\\1&1-t\end{pmatrix},\ \sigma_{31}=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}=e\in S_3. \tag{2.27}$$

The expression for σ_{31} follows from the arrangement of the lexicographic order

$$\begin{pmatrix}12&13\\&23\end{pmatrix},\quad \begin{pmatrix}1&2\\&3\end{pmatrix}\to\begin{pmatrix}1&2\\&3\end{pmatrix}. \tag{2.28}$$

If we compare (2.19) and (2.20) we get

$$k_4^{(t,q)}(\sigma_3)-qt(t-1)E_{14,34}=D_{43}\oplus k_3^{(t,q)}(\sigma_2), \tag{2.29}$$

$$k_4^{(t,q)}(\sigma_2)-qt(t-1)E_{13,23}=D_{42}\oplus k_3^{(t,q)}(\sigma_1), \tag{2.30}$$

$$s_{\sigma_{41}}\big(k_4^{(t,q)}(\sigma_1)-t(t-1)E_{24,12}\big)s_{\sigma_{41}}^{-1}=k_3^{(t,q)}(\sigma_1)\oplus D_{41}, \tag{2.31}$$

$$\text{where}\quad D_{43}=\begin{pmatrix}1&0&0\\0&0&t\\0&1&1-t\end{pmatrix},\ D_{42}=D_{41}=\begin{pmatrix}0&t&0\\1&1-t&0\\0&0&1\end{pmatrix}, \tag{2.32}$$

$$\text{and } \sigma_{41}=\left(\begin{array}{cc|cc|cc}1&2&3&4&5&6\\1&2&4&3&5&6\end{array}\right). \tag{2.33}$$

The expression for σ_{41} follows from the arrangement of the lexicographic order

$$\begin{pmatrix}12&13&14\\&23&24\\&&34\end{pmatrix},\quad \begin{pmatrix}1&2&3\\&4&5\\&&6\end{pmatrix}\to\begin{pmatrix}1&2&4\\&3&5\\&&6\end{pmatrix}. \tag{2.34}$$

If we compare (2.20) and (2.21) we can observe

$$k_5^{(t,q)}(\sigma_4)-qt(t-1)E_{15,45}=D_{54}\oplus k_4^{(t,q)}(\sigma_3), \tag{2.35}$$

$$k_5^{(t,q)}(\sigma_3)-qt(t-1)E_{14,34}=D_{53}\oplus k_3^{(t,q)}(\sigma_2), \tag{2.36}$$

$$k_5^{(t,q)}(\sigma_2)-qt(t-1)E_{13,23}=D_{52}\oplus k_3^{(t,q)}(\sigma_1), \tag{2.37}$$

$$s_{\sigma_{51}}\big(k_5^{(t,q)}(\sigma_1)-t(t-1)E_{25,12}\big)s_{\sigma_{51}}^{-1}=k_3^{(t,q)}(\sigma_1)\oplus D_{51}, \tag{2.38}$$

$$\text{where } \sigma_{51}=\left(\begin{array}{ccc|ccccc|cc}1&2&3&4&5&6&7&8&9&10\\1&2&3&7&4&5&8&6&9&10\end{array}\right)\ \text{and} \tag{2.39}$$

$$D_{54}=\left(\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 0&1&0&t\\0&0&1&1-t\end{array}\right),\ D_{53}=\left(\begin{array}{c|cc|c}1&0&0&0\\\hline 0&0&t&0\\0&1&1-t&0\\\hline 0&0&0&1\end{array}\right),\ D_{52}=D_{51}=\left(\begin{array}{cc|cc}0&t&0&0\\1&1-t&0&0\\\hline 0&0&1&0\\0&0&0&1\end{array}\right). \tag{2.40}$$

The expression for σ_{51} follows from the arrangement of the lexicographic order

$$\begin{pmatrix}12&13&14&15\\&23&24&25\\&&34&35\\&&&45\end{pmatrix},\quad \begin{pmatrix}1&2&3&4\\&5&6&7\\&&8&9\\&&&10\end{pmatrix}\to\begin{pmatrix}1&2&3&7\\&4&5&8\\&&6&9\\&&&10\end{pmatrix}. \tag{2.41}$$

If we compare (2.21) and (2.22) we get connection between $k_6^{(t,q)}$ and $k_5^{(t,q)}$

$$k_6^{(t,q)}(\sigma_5) - qt(t-1)E_{16,56} = D_{65} \oplus k_5^{(t,q)}(\sigma_4), \tag{2.42}$$

$$k_6^{(t,q)}(\sigma_4) - qt(t-1)E_{15,45} = D_{64} \oplus k_5^{(t,q)}(\sigma_3), \tag{2.43}$$

$$k_6^{(t,q)}(\sigma_3) - qt(t-1)E_{14,34} = D_{63} \oplus k_5^{(t,q)}(\sigma_2), \tag{2.44}$$

$$k_6^{(t,q)}(\sigma_2) - qt(t-1)E_{13,23} = D_{62} \oplus k_5^{(t,q)}(\sigma_1), \tag{2.45}$$

$$s_{\sigma_{61}}\big(k_6^{(t,q)}(\sigma_1) - t(t-1)E_{26,12}\big)s_{\sigma_{61}}^{-1} = k_5^{(t,q)}(\sigma_1) \oplus D_{61}, \tag{2.46}$$

$$\text{where } \sigma_{61} = \begin{pmatrix} 1\ 2\ 3\ 4| & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & |14\ 15 \\ 1\ 2\ 3\ 4| & 11 & 5 & 6 & 7 & 12 & 8 & 9 & 13 & 10 & |14\ 15 \end{pmatrix} \text{ and} \tag{2.47}$$

$$D_{65} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 & 1-t \end{array}\right), \quad D_{64} = \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 1-t & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad D_{63} = \left(\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & t & 0 & 0 \\ 0 & 1 & 1-t & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$D_{62} = D_{61} = \left(\begin{array}{cc|ccc} 0 & t & 0 & 0 & 0 \\ 1 & 1-t & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{2.48}$$

The expression for σ_{61} follows from the arrangement of the lexicographic order

$$\begin{pmatrix} 12 & 13 & 14 & 15 & 16 \\ & 23 & 24 & 25 & 16 \\ & & 34 & 35 & 36 \\ & & & 45 & 46 \\ & & & & 56 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & 9 \\ & & 10 & 11 & 12 \\ & & & 13 & 14 \\ & & & & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 11 \\ & 5 & 6 & 7 & 12 \\ & & 8 & 9 & 13 \\ & & & 10 & 14 \\ & & & & 15 \end{pmatrix}. \tag{2.49}$$

The general relations (2.24) we can obtain in a similar way. □

Remark 2.2. We can observe that due to the description of the spectrum $k_n^{(t,q)}$ given below by (2.62), the diagonal part $\mathrm{diag}\big(k_n^{(t,q)}(\sigma_r)\big)$ of $k_n^{(t,q)}(\sigma_r)$ for all $1 \le r \le n$ are the same, up to some permutation:

$$\mathrm{diag}\big(k_n^{t,q}(\sigma_r)\big) = \{qt^2, \underbrace{0,\dots,0}_{n-2 \text{ times}}, \underbrace{1-t,\dots,1-t}_{n-2 \text{ times}}, \underbrace{1,\dots,1}_{(n-1)(n-2)/2-(n-2), \text{ times}}\}. \tag{2.50}$$

2.4. Representations of B_{n+1} via representations of the Lie algebra $\mathfrak{gl}_n$

We can generalize the reduced Burau representation of B_n just observing, see [16], that $\rho_n^{(t)}$ is a product of several $\exp' s$ of an appropriate *Serre generators* of the *natural* (*or fundamental*) *representation* of the Lie algebra $\mathfrak{sl}_n$ (in fact of $\mathfrak{gl}_n$), see (2.57). To be more precise, we show that the reduced Burau representation for B_3

$$\sigma_1 \mapsto \begin{pmatrix} -t & t \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -t \end{pmatrix}, \tag{2.51}$$

can be rewritten as follows:

$$\sigma_1 \mapsto \begin{pmatrix} -t & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \exp(sE_{11})\exp(-E_{12}), \tag{2.52}$$

$$\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix} = \exp(E_{21})\exp(sE_{22}), \tag{2.53}$$

where E_{kn} are *matrix unities* and $s = \ln(-t)$ for negative t.

For B_4 the reduced Burau representation is defined as follows:

$$\rho_4^{(t)} : B_4 \to \mathrm{GL}_3(\mathbb{Z}[t, t^{-1}]),$$

$$\sigma_1 \mapsto \begin{pmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{pmatrix}, \ \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{pmatrix}, \tag{2.54}$$

$$\sigma_1 \mapsto \begin{pmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(sE_{11})\exp(-E_{12}),$$

$$\sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \exp(E_{21})\exp(sE_{22})\exp(-E_{23}),$$

$$\sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -t \end{pmatrix} = \exp(E_{32})\exp(sE_{33}),$$

where $s = \ln(-t)$ when $t < 0$. Finally, we get

$$\sigma_1 = \exp(sE_{11})\exp(-X_1), \ \sigma_2 = \exp(Y_1)\exp(sE_{22})\exp(-X_2), \tag{2.55}$$

$$\sigma_3 = \exp(Y_2)\exp(sE_{33}). \tag{2.56}$$

Remark 2.3. In the general case for arbitrary B_n, the formulas are similar, see (2.57). We can use this formula, factorizations of σ_i, for quantization of the symmetric square of the reduced Burau representations, since it is easy to quantize the Pascal triangle, see Sections 2.10, 2.12, especially formulas (2.106) and (2.108).

Below we give the *Cartan matrix* A corresponding to a Lie algebra $\mathfrak{sl}_n$ and "the rule to obtain" the right formulas, where $X_k = E_{kk+1}, \ Y_k = E_{k+1k}$ and $H_k = E_{kk} - E_{k+1,k+1}$, $1 \le k \le n-1$, are *the Serre generators of* $\mathfrak{sl}_n$:

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} E_{11} & -X_1 & & & & \\ Y_1 & E_{22} & -X_2 & & & \\ & Y_2 & E_{33} & -X_3 & & \\ & & Y_3 & & & \\ & & & \dots & & \\ & & & & -X_{n-2} & \\ & & & & E_{n-1,n-1} & -X_{n-1} \\ & & & & Y_{n-1} & E_{nn} \end{pmatrix}.$$

We put generators $X_k = E_{kk+1}, \ Y_k = E_{k+1k}$ in their appropriate places, corresponding to the indices of E_{kr}.

Lemma 2.2. *Let* $\pi : \mathfrak{gl}_n \to \mathrm{End}(V)$ *be an arbitrary representation of the Lie algebra* $\mathfrak{gl}_n$ *in the space* V*. Then the following formulas*

$$\begin{aligned}&\rho^{\pi}_{n+1}(\sigma_1) \mapsto \exp\big(s\pi(E_{11})\big)\exp\big(-\pi(X_1)\big),\\ &\rho^{\pi}_{n+1}(\sigma_k) \mapsto \exp\big(\pi(Y_{k-1})\big)\exp\big(s\pi(E_{kk})\big)\exp(-\pi(X_k)\big),\\ &\rho^{\pi}_{n+1}(\sigma_n) \mapsto \exp\big(\pi(Y_{n-1})\big)\exp\big(s\pi(E_{nn})\big)\end{aligned} \tag{2.57}$$

give us representation $\rho^{\pi}_{n+1} : B_{n+1} \to \mathrm{GL}(V)$ of the braid group B_{n+1} in the space V. For π being the natural representation *of the Lie algebra $\mathfrak{sl}_n$ we get the reduced Burau representation $\rho^{(t)}_{n+1}$.*

Proof. It is sufficient to use the fact that formulas (2.2) and (2.3) (or (2.4), (2.5)) define representation of the group B_n. □

2.5. Connection between the Lawrence-Krammer and the reduced Burau representation

Theorem 2.3. *The Lawrence-Krammer representation $k^{(t,q)}_n$ defined by (2.16) is equivalent with the* quantization *of the symmetric square of the reduced Burau representation, up to changing of the basis:*

$$k^{(t,q)}_n = s^{-1}_{\sigma_{n1}}\left[S^2(\rho^{(t)}_n)_{e,v}\right]_q s_{\sigma_{n1}}, \tag{2.58}$$

see Remark 2.12 below for the explicit explanation of the notation $\left[S^2(\rho^{(t)}_n)_{e,v}\right]_q$. Here the operator s_σ is defined by (2.23), and σ_{n1} is defined in Lemma 2.1.

Remark 2.4. In any representation of the group B_n all generators σ_k, for $1 \le k \le n-1$, have the same spectrum since they are conjugated:

$$\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}. \tag{2.59}$$

That is why we can speak about the *spectrum of the representation* of B_n.

The spectrum of the reduced Burau representation $\rho^{(t)}_n$ of B_n is as follows:

$$\mathrm{Sp}\,\rho^{(t)}_n(\sigma_k) = \{-t, \underbrace{1,\dots,1}_{n-2\ \text{times}}\}. \tag{2.60}$$

The spectrum of its symmetric square is:

$$\mathrm{Sp}\,S^2(\rho^{(t)}_n(\sigma_k)) = \{t^2, \underbrace{-t,\dots,-t}_{n-2\ \text{times}}, \underbrace{1,\dots,1}_{(n-1)(n-2)/2,\ \text{times}}\}. \tag{2.61}$$

In [26, Claim 5.8] (see also [27, Lemma 5]) the following fact is proved

Lemma 2.4. *The spectrum of the Lawrence-Krammer representation $LK := k^{(t,q)}_n$ of the group B_n, i.e., $\mathrm{Sp}\big(LK(\sigma_k)\big)$ is as follows:*

$$\mathrm{Sp}\big(LK(\sigma_k)\big) = \{qt^2, \underbrace{-t, \dots, -t}_{n-2\ times}, \underbrace{1, \dots, 1}_{(n-1)(n-2)/2,\ times}\}. \tag{2.62}$$

If we set $q = 1$ in (2.62) we get (2.61). Moreover, in [17, Proposition 3.2] the following connection between the Krammer V module and the Burau W module was proved. Fix representation $K_n^{(t,q)}$ defined by (2.15).

Proposition 2.5 *([17]). If $t = 1$ and 2 is invertible in R, then there is an isomorphism of B_n-modules $V \to S^2W$ (symmetric square of W) given by $v_{ij} \mapsto w_{ij}^2$ and, more generally, $v(T) \mapsto w(T)^2$.*

2.6. Summary

1. We don't know how to quantize any representations of the groups B_n.
2. But if the representation is an n-th symmetric power of some representation B_n, then the Pascal triangle appears naturally, see Theorem 2.6 and especially formula (2.71), we know how to quantize the Pascal triangle, see Section 2.9 below.
3. The Lawrence-Krammer (LK) representation $k_n^{(t,q)}$ defined by (2.16) for $q = 1$ is the symmetric square of the Burau representation $S^2(\rho_n^{(t)})$, see Proposition 2.5, that is why we can use the previous item.
4. The fact that all irreducible representations of B_3 in $\dim V \le 5$ obtained by Tuba and Wenzl [28,29] can be given by formulas (2.73), see Theorem 2.8, i.e., by the quantization of the Pascal triangle (up to diagonal matrix Λ_n) means that the q-Pascal triangle reflects the deep properties of the braid groups!
5. Formulas (2.73) cover not only $\dim V \le 5$, but in any dimension $n \in \mathbb{N}$ it gives an $n(n-1)/2$ parameter family of representations of B_3.

2.7. Lie algebra $\mathfrak{sl}_2$ and its finite-dimensional irreducible representations

Recall that the Lie algebra $\mathfrak{s}l_2$ is the Lie algebra of 2×2 real matrices with trace equals to zero. The standard basis X, Y, H in a Lie algebra $\mathfrak{s}l_2$ is as follows:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.63}$$

This natural representations $\pi_2 : \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{C}^2)$ are called *fundamental representation* of the Lie algebra $\mathfrak{s}l_2$.

All finite-dimensional U-module V being the highest weight module of highest weight λ are of the following form see Kassel, [14, Theorem V.4.4.]:

$$\rho(n)(X) = S^n(\pi_2(X)) = \begin{pmatrix} 0 & n & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \rho(n)(Y) = S^n(\pi_2(Y)) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & n & 0 \end{pmatrix}, \tag{2.64}$$

$$\rho(n)(H) = S^n(\pi_2(H)) = \begin{pmatrix} n & 0 & \dots & 0 & 0 \\ 0 & n-2 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & -n+2 & 0 \\ 0 & 0 & \dots & 0 & -n \end{pmatrix}, \tag{2.65}$$

where $\lambda = \dim(V)-1 \in \mathbb{N}$. In fact, all finite-dimensional irreducible representations of $\mathfrak{sl}_2$ are n-symmetric powers of the fundamental representation π_2.

The main idea of the book by Kassel and Turaev [15] is to give a comprehensive introduction to the theory of braid groups.

2.8. Pascal triangle and representations of B_3 and Lie algebra $\mathfrak{sl}_2$

The connection of the Pascal triangle with representations of B_3 was firstly noticed by Humphries in [11]. Set in (2.73) $\Lambda_n = I$ and $q = 1$, then

$$\sigma_1 \mapsto \sigma_1(1,n), \quad \sigma_2 \mapsto \sigma_2(1,n), \quad \text{where} \quad \sigma_2(1,n) = \left(\sigma_1(1,n)^{-1}\right)^\sharp \tag{2.66}$$

is an irreducible representation of the group B_3 in $\mathbb{C}^{n+1}$. Here the *Pascal triangle* denoted by $\sigma_1(1,n)$, for $n-1 \in \mathbb{N}$, is defined as follows:

$$\sigma_1(1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1(1,2) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1(1,3) = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{etc.} \tag{2.67}$$

The central symmetry $A \mapsto A^\sharp$, for the matrix $A \in \mathrm{Mat}(n+1,\mathbb{C})$, $A = (a_{km})_{0\le k,m\le n}$ is defined as follows:

$$A^\sharp = \left(a^\sharp_{km}\right)_{0\le k,m\le n}, \quad a^\sharp_{km} = a_{n-k,n-m}. \tag{2.68}$$

Theorem 2.6. *Let π_2 be the fundamental representation of $\mathfrak{sl}_2$ then*

$$\sigma_1(1,n) = \exp\left(\rho(X)\right) = \exp\left(S^n(\pi_2(X))\right), \tag{2.69}$$
$$\sigma_2(1,n) = \exp\left(\rho(-Y)\right) = \exp\left(S^n(\pi_2(-Y))\right), \tag{2.70}$$

i.e., the Humphries representation of B_3 in dimension $n+1$ is the n-th symmetric power of the Burau representation $\rho_3^{(-1)}$.

As the result of (2.69) we have one nice formula, $\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and in the general case

$$\sigma_1(1,n) = \begin{pmatrix} \binom{n}{n} & \binom{n}{n-1} & \binom{n}{n-2} & \dots & \binom{n}{1} & \binom{n}{0} \\ 0 & \binom{n-1}{n-1} & \binom{n-1}{n-2} & \dots & \binom{n-1}{1} & \binom{n-1}{0} \\ 0 & 0 & \binom{n-2}{n-2} & \dots & \binom{n-2}{1} & \binom{n-2}{0} \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & \binom{1}{1} & \binom{1}{0} \\ 0 & 0 & 0 & \dots & 0 & \binom{0}{0} \end{pmatrix} = \exp\Big(\sum_{k=0}^{n-1}(n-k)E_{k,k+1}\Big). \tag{2.71}$$

2.9. q-Pascal triangle and representations of B_3

We define the family of representations of the group B_3 using *q-Pascal triangle* [1,16]. For $q \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ and a matrix

$$\Lambda_n = \mathrm{diag}(\lambda_r)_{r=0}^n \quad \text{such that} \quad \lambda_r \lambda_{n-r} = const, \quad \text{set} \tag{2.72}$$

$$\sigma_1^{\Lambda_n}(q,n) := \sigma_1(q,n) D_n^\sharp(q) \Lambda_n, \quad \sigma_2^{\Lambda_n}(q,n) := \Lambda_n^\sharp D_n(q) \sigma_2(q,n), \tag{2.73}$$

$$\text{where} \quad D_n(q) = \mathrm{diag}(q_r)_{r=0}^n, \quad q_r = q^{\frac{(r-1)r}{2}}, \tag{2.74}$$

and $\sigma_1(q,n)$ is the *q-Pascal triangle* obtained from the Pascal triangle $\sigma_1(1,n)$ by replacing all natural numbers k by the q-natural numbers $(k)_q$, see (2.79). More precisely, define $\sigma_1(q,n)$ as follows: $\sigma_1(q,n) = \big(\sigma_1(q,n)_{km}\big)_{0\le k,m\le n}$, where

$$\sigma_1(q,n)_{km} = \binom{n-k}{n-m}_q, \ 0 \le k, m \le n, \tag{2.75}$$

$$\sigma_2(q,n) := \big(\sigma_1^{-1}(q^{-1},n)\big)^\sharp, \tag{2.76}$$

$$\big(\sigma_1^{-1}(q^{-1},n)\big)^\sharp_{km} = \begin{cases} 0, & \text{if } 0 < k < m \le n \\ (-1)^{k+m} q_{k-m}^{-1} C_k^m(q^{-1}), & \text{if } 0 \le m \le k \le n \end{cases}. \tag{2.77}$$

Here the *q-binomial coefficients* or *Gaussian polynomials* are defined as follows:

$$C_n^k(q) := \binom{n}{k}_q := \frac{(n)!_q}{(k)!_q (n-k)!_q}, \quad C_n^k[q] := \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q} \tag{2.78}$$

corresponding to the two forms of *q−natural numbers*, defined by

$$(n)_q := \frac{q^n - 1}{q - 1}, \quad [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{2.79}$$

Remark 2.5. We have

$$D_n(q)\sigma_2(q,n) = j_n \big(\sigma_1(q,n)\big)^\sharp j_n^{-1} D_n(q), \ \text{where} \quad j_n = \sum_{r=0}^n (-1)^r E_{r,n-r}. \tag{2.80}$$

Another way to obtain (2.73) is as follows. Define *q-Pochhammer symbol*

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}). \tag{2.81}$$

We have, see [2]:

$$(1+x)_q^k := (-x;q)_k = \sum_{r=0}^k q^{r(r-1)/2} C_k^r(q) x^r = \sum_{r=0}^k q^{r(r-1)/2} \binom{k}{r}_q x^r. \tag{2.82}$$

Therefore, the inversion of the matrix $\sigma_1^{\Lambda_n}(q,n)$ for $\Lambda_n = I_{n+1}$ with respect to the auxiliary diagonal is the matrix of transformations of the monomials $x^0, x^1, x^2, \dots, x^n$ under the rule (2.82).

Theorem 2.7 *([1,16]). Formulas $\sigma_1 \mapsto \sigma_1^{\Lambda_n}(q,n), \quad \sigma_2 \mapsto \sigma_2^{\Lambda_n}(q,n)$, see (2.73), define family of representation of the group B_3 in dimension $n+1$.*

Remark 2.6. Let $\pi : B_3 \to \mathrm{GL}(m,\mathbb{C})$ be a representation. Fix two matrices $\Lambda_1, \Lambda_2 \in \mathrm{GL}(m,\mathbb{C})$ and define $\pi^\Lambda(\sigma_1) = \Lambda_1\pi(\sigma_1), \quad \pi^\Lambda(\sigma_2) = \pi(\sigma_2)\Lambda_2$. When π^Λ is again a representation of B_3? Under the additional conditions $\Lambda_1\Lambda_2 = cI$, $c \in \mathbb{C}$ and $\Lambda_1 = \mathrm{diag}(\lambda_k)_{k=0}^m$ the criterion for π^Λ to be representation is the following:

$$\Lambda_1\pi(\sigma_1\sigma_2\sigma_1) = \pi(\sigma_1\sigma_2\sigma_1)\Lambda_2. \tag{2.83}$$

This implies condition (2.72), i.e., $\lambda_r\lambda_{n-r} = c$.

Theorem 2.8 *([28,1,16]). All representations of the group B_3 in dimension ≤ 5 are given by the formulas (2.73).*

In dimension 2 we get for $\Lambda_1 = \mathrm{diag}(\lambda_0, \lambda_1)$

$$\sigma_1^{\Lambda_1}(q,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad \sigma_2^{\Lambda_1}(q,1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{2.84}$$

In dimension 3, we get for $\Lambda_2 = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2)$ with $\lambda_0\lambda_2 = \lambda_1^2$

$$\sigma_1^{\Lambda_2}(q,2) = \begin{pmatrix} 1 & (1+q) & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} D_2^\sharp(q)\Lambda_2, \quad \sigma_2^{\Lambda_2}(q,2) = \Lambda_2^\sharp \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -(1+q) & 1 \end{pmatrix} D_2(q). \tag{2.85}$$

In dimension 4, we get for $\Lambda_3 = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ with $\lambda_0\lambda_3 = \lambda_1\lambda_2$, see Remark 2.5:

$$\sigma_1^{\Lambda_3}(q,3) = \begin{pmatrix} 1 & (1+q+q^2) & (1+q+q^2) & 1 \\ 0 & 1 & (1+q) & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} D_3^\sharp(q)\Lambda_3, \tag{2.86}$$

$$\sigma_2^{\Lambda_3}(q,3) = \Lambda_3^\sharp \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -(1+q) & 1 & 0 \\ -1 & (1+q+q^2) & -(1+q+q^2) & 1 \end{pmatrix} D_3(q). \tag{2.87}$$

In dimension 5, we get for $\Lambda_4 = \mathrm{diag}(\lambda_r)_{r=0}^4$ with $\lambda_0\lambda_4 = \lambda_1\lambda_3 = \lambda_2^2$:

$$\sigma_1^{\Lambda_4}(q,4) = \begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1 & 1+q & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} D_4^\sharp(q)\Lambda_4, \tag{2.88}$$

$$\sigma_2^{\Lambda_4}(q,4) = \Lambda_4^\sharp \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -(1+q) & 1 & 0 & 0 \\ -1 & (1+q+q^2) & -(1+q+q^2) & 1 & 0 \\ 1 & -(1+q)(1+q^2) & (1+q^2)(1+q+q^2) & -(1+q)(1+q^2) & 1 \end{pmatrix} D_4(q). \tag{2.89}$$

2.10. Quantization of the Pascal triangle

Remark 2.7. The following procedure of quantization ("Quant") is based on the *quantization of the Pascal triangle* explained in [1,16], formulas (10) and (11). To be more precise, we change the binomial coefficients $\binom{n}{k}$ of the Pascal triangle by *q-binomial coefficients* $\binom{n}{k}_q = \frac{(n)!_q}{(k)!_q(n-k)!_q}$.

We introduce the representation $T^{\Lambda_n,q}_{3,n+1}$ of the braid group B_3 equivalent to the representation (2.73) in dimension $n+1$ as follows:

$$T^{\Lambda_n,q}_{3,n+1}(\sigma_1) = \left(\sigma_2^{\Lambda_n}(q,n)\right)^\sharp, \quad T^{\Lambda_n,q}_{3,n+1}(\sigma_2) = \left(\sigma_1^{\Lambda_n}(q,n)\right)^\sharp. \tag{2.90}$$

Lemma 2.9. *We have*

$$T^{\Lambda_n,q}_{3,n+1}(\sigma_1) = J_n\sigma_2^{\Lambda_n}(q,n)J_n^{-1}, \quad T^{\Lambda_n,q}_{3,n+1}(\sigma_2) = J_n\sigma_1^{\Lambda_n}(q,n)J_n^{-1}. \tag{2.91}$$

Proof. We use the fact that $A^\sharp = J_nAJ_n^{-1}$, where $J_n = \sum_{r=0}^n E_{r,n-r}$. $\square$

Remark 2.8. Using [29], see also [16], we conclude that all representation of B_3 in dimension $2, 3, 4$ and 5 are equivalent with the following ones, where $\Lambda_1 = s\,\mathrm{diag}(-t,1)$ and $s,t \in \mathbb{C}^\times$:

$$T^{\Lambda_1,1}_{32}(\sigma_1) = s\begin{pmatrix} -t & t \\ 0 & 1\end{pmatrix} = s\begin{pmatrix} -t & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}, \; T^{\Lambda_1,1}_{32}(\sigma_2) = s\begin{pmatrix} 1 & 0 \\ 1 & -t\end{pmatrix} = s\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -t\end{pmatrix}. \tag{2.92}$$

All representations of B_3 in dimension 3 are given by the following formulas, where $s\Lambda_2 = s\mathrm{diag}(t^2,-t,1) = s\mathrm{diag}(t^2,-t,1)$ and $s,t,q \in \mathbb{C}^\times$:

$$T^{\Lambda_2,q}_{33}(\sigma_1) = s\begin{pmatrix} t^2q & -t^2(1+q) & t^2 \\ 0 & -t & t \\ 0 & 0 & 1\end{pmatrix} = s\Lambda_2\begin{pmatrix} 1 & -(1+q) & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1\end{pmatrix} D_2^\sharp(q), \tag{2.93}$$

$$T^{\Lambda_2,q}_{33}(\sigma_2) = s\begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & 0 \\ 1 & -t(1+q) & t^2q\end{pmatrix} = s\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & (1+q) & 1\end{pmatrix} D_2(q)\Lambda_2^\sharp. \tag{2.94}$$

Remark 2.9. If we are sufficiently attentive, we can recognize here the Lawrence-Krammer representation of B_3 for $s = 1$, see formulas (2.109) and (2.116) below.

All representations of B_3 in dimension 4 are given by the following formulas, where $\Lambda_3 = s\mathrm{diag}\,(-t^3, ut^2, -u^{-1}t, t)$:

$$T^{\Lambda_3,q}_{34}(\sigma_1) = \Lambda_3\begin{pmatrix} 1 & -(1+q+q^2) & (1+q+q^2) & -1 \\ 0 & 1 & -(1+q) & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1\end{pmatrix} D_3^\sharp(q), \tag{2.95}$$

$$T^{\Lambda_3,q}_{34}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & (1+q) & 1 & 0 \\ 1 & (1+q+q^2) & (1+q+q^2) & 1\end{pmatrix} D_3(q)\Lambda_3^\sharp. \tag{2.96}$$

For $n = 5$ we get

$$T_{35}^{\Lambda_4,q}(\sigma_1) = \Lambda_4 \begin{pmatrix} 1 & -(1+q)(1+q^2) & (1+q^2)(1+q+q^2) & -(1+q)(1+q^2) & -1 \\ 0 & 1 & -(1+q+q^2) & (1+q+q^2) & -1 \\ 0 & 0 & 1 & -(1+q) & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} D_4^\sharp(q), \tag{2.97}$$

$$T_{35}^{\Lambda_4,q}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & (1+q) & 1 & 0 & 0 \\ 1 & (1+q+q^2) & (1+q+q^2) & 1 & 0 \\ 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \end{pmatrix} D_4(q)\Lambda_4^\sharp. \tag{2.98}$$

Remark 2.10. The irreducibility criteria for the representation (2.73) of B_3 in dimensions $n \le 5$ were obtained in [28,29].

Remark 2.11. For dimensions $n \ge 6$ formulas (2.73) gives us some family of representations of B_3 but not all, see [29, Remark 2.11.3]. For more representations of B_3 see, e.g., Westbury, [30]. The question of the irreducibility of representations (2.73) is open for $n \ge 6$.

2.11. Symmetric basis in $V \otimes V$

Let V be a finite-dimensional linear space with the basis $(e_k)_{k=1}^n$. The *tensor product* $V \otimes V$ is a linear space generated by the basis $\big(e_k \otimes e_r\big)_{k,r=1}^n$. The *symmetric square* $S^2(V)$ of the space V is a subspace of $V \otimes V$ generated by one of the equivalent *symmetric basis* $e = \big(e_{kn}^s\big)_{kn}$ or $v = \big(v_{kn}\big)_{kn}$, defined as follows:

$$e_{kk}^s = e_k \otimes e_k,\ 1 \le k \le n, \quad e_{kr}^s = e_k \otimes e_r + e_r \otimes e_k,\ 1 \le k < r \le n, \tag{2.99}$$

$$v_{kr} = (e_k + \cdots + e_{r-1}) \otimes (e_k + \cdots + e_{r-1}), \quad 1 \le k < r \le n. \tag{2.100}$$

We fix a *lexicographic order* on the set (k, r) with $1 \le k \le r \le n$. Let an operator A acts on a space V. Denote by $A \otimes A$ its tensor product on a space $V \otimes V$ and by $S^2(A)$ its *symmetric square.*

2.12. Proof of Theorem 2.3

Remark 2.12. We show that $\big[S^2(\rho_n^{(t)})_{e,v}\big]_q$, the quantization $[\dots]_q$ of the symmetric square $S^2(\rho_n^{(t)})_e$ of the reduced Burau representations $\rho_n^{(t)}$ for the group B_n, expressed in the basis e defined by (2.99) and *calculated further in the basis* v defined by (2.100), coincide with the Lawrence-Krammer representation $k_n^{(t,q)}$ in notations (2.16), i.e., we show that

$$s_{\sigma_{n1}}^{-1} C_n \big[S^2\big(\rho_n^{(t)}(\sigma_r)\big)_e\big]_q C_n^{-1} s_{\sigma_{n1}} = k_n^{(t,q)}(\sigma_r), \quad \text{for} \quad 1 \le r \le n-1, \tag{2.101}$$

where C_n is the *change-of-basis matrix* from the basis e defined by (2.99) to the basis v defined by (2.100) for the vector space $S^2(\mathbb{C}^{n-1})$. Here the operator s_σ is defined by (2.23), and σ_{n1} is defined in Lemma 2.1.

The Burau representation $\rho_3^{(t)}$ for B_3 on the space $V = \mathbb{C}^2$ is as follows:

$$\sigma_1 \mapsto \begin{pmatrix} -t & t \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -t \end{pmatrix}. \tag{2.102}$$

We fix the standard basis $e = (e_k)_{k=1}^n$ in $\mathbb{C}^n$, namely set

$$e_k = (\underbrace{0, \dots, 0, 1}_{k \text{ place}}, 0, \dots, 0), \quad 1 \le k \le n. \tag{2.103}$$

In the symmetric square $S^2(\mathbb{C}^2)$ the basis e and v are the following:

$$e_{11}^s = e_1 \otimes e_1, \quad e_{12}^s = e_1 \otimes e_2 + e_2 \otimes e_1, \quad e_{22}^s = e_2 \otimes e_2, \tag{2.104}$$

$$v_{12} = e_1 \otimes e_1, \quad v_{13}^s = (e_1 + e_2) \otimes (e_1 + e_2), \quad v_{23} = e_2 \otimes e_2. \tag{2.105}$$

The *quantization* of the symmetric square $\left[S^2(\sigma_k)_e\right]_q := \left[S^2\left(\rho_3^{(t)}(\sigma_k)\right)_e\right]_q$ for $k = 1, 2$ of the Burau representation $\rho_3^{(t)}$ has the following form in the basis e:

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \overset{\mathrm{Sym}^2}{\mapsto} \begin{pmatrix} t^2 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mathrm{Quant}}{\mapsto}$$
$$\begin{pmatrix} t^2 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -(1+q) & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t^2 q & -t^2(1+q) & t^2 \\ 0 & -t & t \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.106}$$

We denote

$$D_{1,2}(q) = \mathrm{diag}(q, 1, 1) \quad \text{and} \quad D_{2,2}(q) = \mathrm{diag}(1, 1, q). \tag{2.107}$$

Similarly, we get

$$\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix} \overset{\mathrm{Sym}^2}{\mapsto} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & t^2 \end{pmatrix} \overset{\mathrm{Quant}}{\mapsto}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & (1+q) & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & 0 \\ 1 & -t(1+q) & t^2 q \end{pmatrix}. \tag{2.108}$$

Finally, we have

$$\left[S^2(\sigma_1)_e\right]_q := \begin{pmatrix} t^2 q & -t^2(1+q) & t^2 \\ 0 & -t & t \\ 0 & 0 & 1 \end{pmatrix}, \quad \left[S^2(\sigma_2)_e\right]_q := \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & 0 \\ 1 & -t(1+q) & t^2 q \end{pmatrix}. \tag{2.109}$$

To enumerate the elements of the basis e and v by the same set of indices, set

$$w_{ij} = v_{i,j-1}. \tag{2.110}$$

Since the basis e and w (hence v) are connected as follows:

$$\begin{array}{llll} e^s_{11} = & w_{11} & & \\ e^s_{12} = & -w_{11} & +w_{12} & -w_{22}\,, \\ e^s_{22} = & & & w_{22} \end{array} \qquad \begin{array}{llll} w_{11} = & e^s_{11} & & \\ w_{12} = & e^s_{11} & +e^s_{12} & +e^s_{22}\,, \\ w_{22} = & & & e^s_{22} \end{array} \tag{2.111}$$

the *change-of-basis matrix* C_3 in the space $S^2(\mathbb{C}^2)$ will be the following:

$$C_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad C_3^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{2.112}$$

We show that

$$C_3 \big[S^2(\sigma_r)_e\big]_q C_3^{-1} = k_3^{(t,q)}(\sigma_r), \quad \text{for} \quad r = 1, 2. \tag{2.113}$$

Here by (2.26) we have $\sigma_{31} = e \in S_3$, that is why $s_{\sigma_{31}} = I$. Indeed, we have

$$[S^2(\sigma_1)_{e,v}]_q := C_3 \begin{pmatrix} t^2q & -t^2(1+q) & t^2 \\ 0 & -t & t \\ 0 & 0 & 1 \end{pmatrix} C_3^{-1} = \begin{pmatrix} t^2q & 0 & t(t-1) \\ 0 & 0 & t \\ 0 & 1 & 1-t \end{pmatrix}, \tag{2.114}$$

$$[S^2(\sigma_2)_{e,v}]_q := C_3 \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & 0 \\ 1 & -t(1+q) & t^2q \end{pmatrix} C_3^{-1} = \begin{pmatrix} 0 & t & 0 \\ 1 & 1-t & 0 \\ 0 & qt(t-1) & t^2q \end{pmatrix}. \tag{2.115}$$

This coincides with *the Krammer representation* $k_3^{(t,q)}$ *for* B_3 in notations (2.16):

$$k_3^{(t,q)}(\sigma_1) = \begin{pmatrix} t^2q & 0 & t(t-1) \\ 0 & 0 & t \\ 0 & 1 & 1-t \end{pmatrix}, \quad k_3^{(t,q)}(\sigma_2) = \begin{pmatrix} 0 & t & 0 \\ 1 & 1-t & 0 \\ 0 & qt(t-1) & t^2q \end{pmatrix}. \tag{2.116}$$

The quantization of the symmetric square $S^2(\rho^{(t)}_{n+1})$ of the Burau representations $\rho^{(t)}_{n+1}$ for the group B_{n+1} will be defined as follows, compare with (2.57):

$$\sigma_1 \to [S^2(\sigma_1)]_q = S^2\big(\exp(sE_{11})\big)\big[S^2\big(\exp(-E_{12})\big)\big]_q D_{1,n}(q), \tag{2.117}$$

$$\sigma_n \to [S^2\big(\exp(E_{n-1,n})\big)]_q D_{n,n}(q) S^2\big(\exp(sE_{nn})\big), \tag{2.118}$$

$$\sigma_k \to \big[S^2\big(\exp(E_{k,k-1})\big)\big]_q D_{k,n}(q) S^2\big(\exp(sE_{kk})\big)\big[S^2\big(\exp(-E_{kk+1})\big]_q, \tag{2.119}$$

where $D_{k,n}(q)$ is defined for $1 \le k \le n$ and $1 \le r \le l \le n$ as follows:

$$D_{k,n}(q)e^s_{r,l} = qe^s_{r,l} \quad \text{if} \quad k = r = l, \quad D_{k,n}(q)e^s_{r,l} = e^s_{r,l}, \quad \text{otherwise}. \tag{2.120}$$

The Burau representation $\rho_4^{(t)}$ for B_4 is as follows:

$$\sigma_1 \mapsto \begin{pmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{pmatrix}. \tag{2.121}$$

The symmetric square of the reduced Burau representation $\rho_4^{(t)}$ for B_4 is as follows, see (2.54) and (2.55):

$$S^2(\sigma_1) \mapsto \begin{pmatrix} t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t^2 & -2t^2 & t^2 & 0 & 0 & 0 \\ 0 & -t & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S^2(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -t & 0 & t & 0 & 0 \\ 1 & -2t & t^2 & 2t & -2t^2 & t^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S^2(\sigma_3) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & -2t & t^2 \end{pmatrix}.$$

Using formulas (2.117), (2.118) and (2.119) we get

$$S^2(\sigma_1) \overset{\text{Quant}}{\mapsto} \begin{pmatrix} t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q & -(1+q) & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t^2 q & -t^2(1+q) & t^2 & 0 & 0 & 0 \\ 0 & -t & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S^2(\sigma_3) \overset{\text{Quant}}{\mapsto} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & (1+q) & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & -t(1+q) & t^2 q \end{pmatrix}.$$

$$S^2(\sigma_2) \overset{\text{Quant}}{\mapsto} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1+q & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & qt^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -(1+q) & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -t & 0 & 0 & 0 & 0 \\ 1 & -t(1+q) & qt^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -(1+q) & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -t & 0 & t & 0 & 0 \\ 1 & -t(1+q) & t^2 q & t(1+q) & -t^2(1+q) & t^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, we get in the basis e:

$$\left[S^2(\sigma_1)_e\right]_q := \begin{pmatrix} t^2 q & -t^2(1+q) & t^2 & 0 & 0 & 0 \\ 0 & -t & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \left[S^2(\sigma_3)_e\right]_q := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & -t(1+q) & t^2 q \end{pmatrix}, \tag{2.122}$$

$$\left[S^2(\sigma_2)_e\right]_q := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & -t & 0 & t & 0 & 0\\ 1 & -t(1+q) & t^2q & t(1+q) & -t^2(1+q) & t^2\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -t & t\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.123}$$

We rewrite (2.122) and (2.123) in the basis v and compare with the expressions $k_4^{(t,q)}(\sigma_r)$, for $r=1,2,3$. We keep the following order to enumerate the elements of the basis e and w in $S^2(\mathbb{C}^3)$:

$$e = (e^s_{11}, e^s_{12}, e^s_{22}, e^s_{13}, e^s_{23}, e^s_{33}), \quad w = (w_{11}, w_{12}, w_{22}, w_{13}, w_{23}, w_{33}). \tag{2.124}$$

By Lemma 2.10, the change-of-basis matrices C_4 and C_4^{-1} in the space $S^2(\mathbb{C}^4)$ are the following:

$$C_4 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0\\ 0 & -1 & 1 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad C_4^{-1} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{2.125}$$

We calculate $\left[S^2(\sigma_r)_v\right]_q = C_4\left[S^2(\sigma_r)_e\right]_q C_4^{-1}$ for $r = 1, 2, 3$. We have

$$\left[S^2(\sigma_1)_v\right]_q = C_4 \begin{pmatrix} t^2q & -t^2(1+q) & t^2 & 0 & 0 & 0\\ 0 & -t & t & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -t & t & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} C_4^{-1} = \begin{pmatrix} qt^2 & 0 & t(t-1) & 0 & t(t-1) & 0\\ 0 & 0 & t & 0 & 0 & 0\\ 0 & 1 & 1-t & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & t & 0\\ 0 & 0 & 0 & 1 & 1-t & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.126}$$

$$\left[S^2(\sigma_2)_v\right]_q = C_4 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & -t & 0 & t & 0 & 0\\ 1 & -t(1+q) & t^2q & t(1+q) & -t^2(1+q) & t^2\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -t & t\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} C_4^{-1} = \begin{pmatrix} 0 & t & 0 & 0 & 0 & 0\\ 1 & 1-t & 0 & 0 & 0 & 0\\ 0 & qt(t-1) & qt^2 & 0 & 0 & t(t-1)\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0 & 1 & 1-t \end{pmatrix},$$

$$\left[S^2(\sigma_3)_v\right]_q = C_4 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -t & 0 & 0\\ 0 & 0 & 1 & 0 & -t & 0\\ 0 & 0 & 1 & 0 & -t(1+q) & t^2q \end{pmatrix} C_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & t & 0 & 0\\ 0 & 1 & 0 & 0 & t & 0\\ 0 & 0 & 1 & 1-t & 0 & 0\\ 0 & 0 & 0 & 0 & 1-t & 0\\ 0 & 0 & 0 & qt(t-1) & qt(t-1) & qt^2 \end{pmatrix}. \tag{2.127}$$

Applying the operator $s_{\sigma_{41}}$ defined by (2.23), where σ_{41} is defined by (2.33), see (2.124), we get $s^{-1}_{\sigma_{41}}\left[S^2(\sigma_r)_v\right]_q s_{\sigma_{41}} = k_4^{(t,q)}(\sigma_r)$ for $1 \le r \le 3$. This coincides with *the LK representation $k_4^{(t,q)}(\sigma_r)$ for B_4* in notations (2.16):

$$k_4^{(t,q)}(\sigma_1) = \begin{pmatrix} qt^2 & 0 & 0 & t(t-1) & t(t-1) & 0\\ 0 & 0 & 0 & t & 0 & 0\\ 0 & 0 & 0 & 0 & t & 0\\ 0 & 1 & 0 & 1-t & 0 & 0\\ 0 & 0 & 1 & 0 & 1-t & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad k_4^{(t,q)}(\sigma_2) = \begin{pmatrix} 0 & t & 0 & 0 & 0 & 0\\ 1 & 1-t & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & qt(t-1) & 0 & qt^2 & 0 & t(t-1)\\ 0 & 0 & 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0 & 1 & 1-t \end{pmatrix},$$

$$k_4^{(t,q)}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 1-t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 1-t & 0 \\ 0 & 0 & qt(t-1) & 0 & qt(t-1) & qt^2 \end{pmatrix}.$$

Remark 2.13. Rinat Kashaev pointed out to the author during his visit to Geneva in May 2022, that the formulas (2.126)–(2.127) should be corrected.

Lemma 2.10. *The matrices C_{n+1}^{-1} and C_{n+1} are defined as follows. The columns of the matrix C_{n+1}^{-1} are defined by the following relations:*

$$w_{ij} = \sum_{i\le k\le r\le j} e_{kr}^s, \quad 1 \le i \le j \le n. \tag{2.128}$$

The columns of the matrix C_{n+1} are defined by the following relations:

$$e_{ij}^s = \begin{cases} (-1)^{i+j}\sum_{i\le k\le r\le j}(-1)^{k+r}w_{kr}, & \text{if } 0 \le j-i \le 1, \\ \sum_{r=i+1}^{j-1}(-w_{ir}+w_{i+1,r}) + w_{ij} - w_{i+1,j}, & \text{if } i=1, \quad 2 \le j-i, \\ -w_{ii} + \sum_{r=i+1}^{j-1}(-w_{ir}+w_{i+1,r}) + w_{ij} - w_{i+1,j}, & \text{if } i>1, \quad 2\le j-i. \end{cases} \tag{2.129}$$

Proof. Indeed, to prove (2.112), (2.125) and (2.133) we note that matrix $E_n = C_n^{-1}$ is block upper-triangular

$$E_n = \begin{pmatrix} a_n & b_n \\ 0 & c_n \end{pmatrix}, \quad \text{therefore} \quad C_n = E_n^{-1} = \begin{pmatrix} a_n^{-1} & -a_n^{-1}b_nc_n^{-1} \\ 0 & c_n^{-1} \end{pmatrix}. \tag{2.130}$$

For $n = 3$ we have

$$E_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} a_3^{-1} & -a_3^{-1}b_3c_3^{-1} \\ 0 & c_3^{-1} \end{pmatrix},$$

where

$$a_3 = (1), \quad b_3 = (1\ 0), \quad c_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We have $c_3^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, therefore $a_3^{-1}b_3c_3^{-1} = (1)(1\ 0)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (1\ 0)$, hence $C_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. For $n = 4$ we get

$$E_4 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a_4 & b_4 \\ 0 & c_4 \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} a_4^{-1} & -a_4^{-1}b_4c_4^{-1} \\ 0 & c_4^{-1} \end{pmatrix}$$

where

$$a_4 = E_3, \quad b_4 = \begin{pmatrix} 1&0&0\\ 1&0&0\\ 1&1&0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 1&0&0\\ 1&1&0\\ 1&1&1 \end{pmatrix}.$$

Since $c_4^{-1} = \begin{pmatrix} 1&0&0\\ -1&1&0\\ 0&-1&1 \end{pmatrix}$ we get

$$a_4^{-1}b_4c_4^{-1} = \begin{pmatrix} 1&-1&0\\ 0&1&0\\ 0&-1&1 \end{pmatrix}\begin{pmatrix} 1&0&0\\ 1&0&0\\ 1&1&1 \end{pmatrix}\begin{pmatrix} 1&0&0\\ -1&1&0\\ 0&-1&1 \end{pmatrix} = \begin{pmatrix} 0&0&0\\ 1&0&0\\ -1&1&0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1&-1&0&0&0&0\\ 0&1&0&-1&0&0\\ 0&-1&1&1&-1&0\\ 0&0&0&1&0&0\\ 0&0&0&-1&1&0\\ 0&0&0&0&-1&1 \end{pmatrix}.$$

For $n = 5$ we have

$$E_5 = C_5^{-1} = \left(\begin{array}{c|cc|ccc|cccc} 1&1&0&1&0&0&1&0&0&0\\ \hline 0&1&0&1&0&0&1&0&0&0\\ 0&1&1&1&1&0&1&1&0&0\\ \hline 0&0&0&1&0&0&1&0&0&0\\ 0&0&0&1&1&0&1&1&0&0\\ 0&0&0&1&1&1&1&1&1&0\\ \hline 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&1&1&1&1 \end{array}\right) = \begin{pmatrix} a_5 & b_5\\ 0 & c_5 \end{pmatrix}, \quad E_5^{-1} = \begin{pmatrix} a_5^{-1} & -a_5^{-1}b_5c_5^{-1}\\ 0 & c_5^{-1} \end{pmatrix},$$

where

$$a_5 = E_4, \quad b_5 = \begin{pmatrix} 1&0&0&0\\ 1&0&0&0\\ 1&1&0&0\\ 1&0&0&0\\ 1&1&0&0\\ 1&1&1&0 \end{pmatrix}, \quad c_5 = \begin{pmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&1&1&0\\ 1&1&1&1 \end{pmatrix}.$$

Since $c_5^{-1} = \begin{pmatrix} 1&0&0&0\\ -1&1&0&0\\ 0&-1&1&0\\ 0&0&-1&1 \end{pmatrix}$ we get

$$a_5^{-1}b_5c_5^{-1} = \begin{pmatrix} 1&-1&0&0&0&0\\ 0&1&0&-1&0&0\\ 0&-1&1&1&-1&0\\ 0&0&0&1&0&0\\ 0&0&0&-1&1&0\\ 0&0&0&0&-1&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ 1&0&0&0\\ 1&1&0&0\\ 1&0&0&0\\ 1&1&0&0\\ 1&1&1&0 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ -1&1&0&0\\ 0&-1&1&0\\ 0&0&-1&1 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ -1&1&0&0\\ 1&0&0&0\\ -1&1&0&0\\ 0&-1&1&0 \end{pmatrix},$$

and

$$C_5 = \left(\begin{array}{c|cc|ccc|cccc} 1&-1&0&0&0&0&0&0&0&0\\ \hline 0&1&0&-1&0&0&-1&0&0&0\\ 0&-1&1&1&-1&0&1&-1&0&0\\ \hline 0&0&0&1&0&0&-1&0&0&0\\ 0&0&0&-1&1&0&1&-1&0&0\\ 0&0&0&0&-1&1&0&1&-1&0\\ \hline 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&-1&1&0&0\\ 0&0&0&0&0&0&0&-1&1&0\\ 0&0&0&0&0&0&0&0&-1&1 \end{array}\right). \quad \square$$

Remark 2.14. The matrix $E_n = C_n^{-1}$ and $E_n^{-1} = C_n$ have the following form

$$E_n = \begin{pmatrix} e_{11} & e_{12} & e_{13} & \dots & e_{1n-1} \\ 0 & e_{22} & e_{23} & \dots & e_{2n-1} \\ 0 & 0 & e_{33} & \dots & e_{3n-1} \\ & & & \dots & \\ 0 & 0 & 0 & \dots & e_{n-1,n-1} \end{pmatrix}, \quad E_n^{-1} = \begin{pmatrix} e_{11}^{-1} & e_{12}^{-1} & e_{13}^{-1} & \dots & e_{1n-1}^{-1} \\ 0 & e_{22}^{-1} & e_{23}^{-1} & \dots & e_{2n-1}^{-1} \\ 0 & 0 & e_{33}^{-1} & \dots & e_{3n-1}^{-1} \\ & & & \dots & \\ 0 & 0 & 0 & \dots & e_{n-1,n-1}^{-1} \end{pmatrix}, \tag{2.131}$$

where $e_{kr} \in \mathrm{Mat}(k \times r, \mathbb{C})$ for $1 \le k \le r \le n-1$ are as follows:

$$e_{11} = (1),\ e_{22} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right),\ e_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},\ e_{kk} = (I - e_k)^{-1},\ e_k = \sum_{r=1}^{k-1} E_{r+1,r},$$

$e_{kr} = (e_{kk}, 0_{k,r-k})$, where $0_{k,r} = 0$ in $\mathrm{Mat}(k \times r, \mathbb{C})$. For E_n^{-1} we have

$$e_{11}^{-1} = (1),\ e_{22}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ e_{33}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},\ e_{kk} = (I - e_k),$$

and $e_{1r}^{-1} = 0$ for $2 \le r \le n-1$, $e_{kr}^{-1} := (-e_{kk}^{-1}, 0_{k,r-k})$ for $2 \le k < r \le n-1$. Notation e_{kr}^{-1} for $k < r$ does not mean the inverse matrix to e_{kr}!

The Burau representation $\rho_5^{(t)}$ for B_5 is as follows:

$$\sigma_1 \mapsto \begin{pmatrix} -t & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -t & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -t & t \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \sigma_4 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -t \end{pmatrix}. \tag{2.132}$$

By Lemma 2.10, the change-of-basis matrix C_5 and C_5^{-1} in the space $S^2(\mathbb{C}^5)$ are the following:

$$C_5 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad C_5^{-1} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.133}$$

Using (2.16) we get

$$k_5^{(t,q)}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & qt(t-1) & 0 & 0 & qt(t-1) & 0 & qt^2 & 0 & t(t-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1-t \end{pmatrix}.$$

Indeed, since

$$\sigma_3 \mapsto \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&-t&t\\ 0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-t&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&-1\\ 0&0&0&1 \end{pmatrix} =$$

$$\exp(E_{32})\exp(sE_{33})\exp(-E_{34}),$$

and

$$S^2(e^{E_{32}}) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0\\ 0&0&1&0&2&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad S^2(e^{-E_{34}}) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0&-1&0&0\\ 0&0&0&0&0&1&0&0&-2&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

we get

$$S^2(\sigma_3) \overset{\text{Quant}}{\mapsto}$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0\\ 0&0&1&0&(1+q)&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&-t&0&0&1&0&0&0\\ 0&0&0&0&-t&0&0&0&0&0\\ 0&0&0&0&0&qt^2&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&-t&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0&-1&0&0\\ 0&0&0&0&0&1&0&0&-(1+q)&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&0&0&-t&0&0&0&0&0&0\\ 0&0&1&0&-t&0&0&0&0&0\\ 0&0&1&0&-t(1+q)&qt^2&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&-t&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0&-1&0&0\\ 0&0&0&0&0&1&0&0&-(1+q)&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&t&1&0&0&0&0&0&0&0\\ 0&0&0&-t&0&0&t&0&0&0\\ 0&0&1&0&-t&0&0&t&0&0\\ 0&0&1&0&-t(1+q)&qt^2&0&t(1+q)&-t^2(1+q)&t^2\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&-t&t\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Applying the operator $s_{\sigma_{51}}$ defined by (2.23), where σ_{51} is defined by (2.33) we get $s_{\sigma_{51}}^{-1} C_5 \big[S^2(\sigma_3)_v\big]_q C_5^{-1} s_{\sigma_{51}} = k_5^{(t,q)}(\sigma_3)$. We use the order in the basis $S^2(\mathbb{C}^4)$ like in (2.124). Indeed,

$$s_{\sigma_{51}}^{-1} C_5 \left[S^2(\rho_5^{(t)}(\sigma_3)_e\right]_q C_5^{-1} s_{\sigma_{51}} =$$

$$s_{\sigma_{51}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -t & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & -t(1+q) & qt^2 & 0 & t(1+q) & -t^2(1+q) & t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} s_{\sigma_{51}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & qt(t-1) & 0 & 0 & t(t-1) & 0 & qt^2 & 0 & t(t-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1-t \end{pmatrix} \stackrel{(2.16)}{=} k_5^{(t,q)}(\sigma_3).$$

The formulas (2.21) can be obtained similarly.

Remark 2.15. We can prove (2.101) for general B_n. The reason is the following. By Remark 2.1 we can construct the reduced Burau representation for B_{n+1} from the reduced Burau representation for B_n using formula (2.13). In particular, we can construct the reduced Burau representation for B_5 starting from the reduced Burau representation for B_4 by formulas (2.14).

By Lemma 2.1 we can reconstruct the LK representation $k_{n+1}^{(t,q)}$ for B_{n+1} from the LK representation $k_n^{(t,q)}$ for B_n defined by (2.16). That is why it is sufficient to prove Theorem 2.3 only for B_3 and B_4.

3. How to construct new representations of B_n

3.1. Representations of a Lie algebra $\mathfrak{sl}_{n+1}$

We recall, following to J.P. Serre [25], Part I, Ch. VII, §1; Part III, Ch. VII, §6, that all irreducible finite-dimensional representations of a *simple* Lie algebra $\mathfrak{sl}_{n+1}$ are *highest weight representations* and are contained in tensor powers of the *natural representation* π_{n+1} [31].

The *Cartan subalgebra* $\mathfrak{h}$ of the Lie algebra $\mathfrak{sl}_{n+1}$ consists of all matrices of the form $H = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$ with $\sum_k \lambda_k = 0$. The *roots* of a Lie algebra $\mathfrak{sl}_{n+1}$ are linear functionals $\alpha_{i,j} \in \mathfrak{h}^*$, $i \neq j$, of the form

$$\alpha_{i,j}(H) = \lambda_i - \lambda_j, \quad \text{where} \quad H = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1}).$$

The *fundamental wights* ω_k are defined by

$$\omega_k(H) = \lambda_1 + \cdots + \lambda_k, \quad \text{where} \quad H = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1}).$$

The fundamental weight ω_1 is the fundamental weight of the *standard representation* π_{n+1} of the Lie algebra $\mathfrak{sl}_{n+1}$ in $E=\mathbb{C}^{n+1}$. Moreover, the fundamental weight ω_k is the highest weight of the representation $\wedge^k(\pi_{n+1})$ of $\mathfrak{sl}_{n+1}$ in the space $\wedge^k E$ (see [25, 1.7.4.6]), where $\wedge^k(A)$ is k^{th} *exterior power* of A. In the particular case, all irreducible representations of $\mathfrak{sl}_2$ are $\mathrm{Sym}^n(\pi_2)$ in $\mathrm{Sym}^n(\mathbb{C}^2)$, $n\in\mathbb{N}$. For $\mathfrak{sl}_3$ all irreducible representations are contained in tensor product of π_3 and $\wedge^2(\pi_3)$, corresponding to fundamental weights ω_1 and ω_2. All irreducible representations of $\mathfrak{sl}_{n+1}$ are contained in tensor products of $\wedge^k(\pi_{n+1})$, $1\le k\le n$.

3.2. New representations of B_n

Let $T_{nm}:\mathfrak{sl}_n\to \mathrm{End}(\mathbb{C}^m)$ be any finite-dimensional representation of a Lie algebra $\mathfrak{sl}_n$ then $\exp(T_{nm}):\mathrm{SL}(n,\mathbb{C})\to\mathrm{GL}(m,\mathbb{C})$ is representation of $\mathrm{SL}(n,\mathbb{C})$. We can define two representations of B_{n+1} as follows

$$\sigma_k\to S^r\big(\exp(T_{nm})\circ\rho_n^{(t)}(\sigma_k)\big),\quad \sigma_k\to\wedge^l\big(\exp(T_{nm})\circ\rho_n^{(t)}(\sigma_k)\big),\tag{3.1}$$

for $r\ge 2$ and $2\le l\le m-1$. Consider the exterior square $\wedge^2(\rho_{2,4}^{(t)})$ of the Burau representation $\rho_{2,4}^{(t)}$ of the group B_4 defined by (2.4) and (2.5) in the basis $e_{ij}^{\wedge}$, $1\le i<j\le 3$ of the space $\wedge^2(\mathbb{C}^3)$

$$e_{12}^{\wedge}=e_1\otimes e_2-e_2\otimes e_1,\quad e_{13}^{\wedge}=e_1\otimes e_3-e_3\otimes e_1,\quad e_{23}^{\wedge}=e_2\otimes e_3-e_3\otimes e_2.$$

Lemma 3.1. *Representation $\wedge^2(\rho_{2,4}^{(t)})$ of the group B_4 is defined by*

$$\sigma_1\to\begin{pmatrix}-t&0&0\\0&-t&t\\0&0&1\end{pmatrix},\quad \sigma_2\to\begin{pmatrix}-t&t&0\\0&1&0\\0&1&-t\end{pmatrix},\quad \sigma_3\to\begin{pmatrix}1&0&0\\1&-t&0\\0&0&-t\end{pmatrix}.\tag{3.2}$$

We have

$$\wedge^2\rho_{2,4}^{(t)}\not\sim\rho_4^{(t)},\quad \textit{for } t\neq -1,\quad \textit{but}\ \ \wedge^2\rho_{2,4}^{(t)}=-tJ_2\rho_{1,4}^{(-t^{-1})}J_2^{-1},\tag{3.3}$$

where $\rho_{1,4}^{(t)}$ is defined by (2.2) and (2.3) and $J_2=\sum_{k=1}^3 E_{k,4-k}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$.

Proof. The first part of (3.3) follows from comparison of two spectrums: $\mathrm{Sp}\,\rho_{2,4}^{(t)}(\sigma_1)=\{-t,1,1\}$ and $\mathrm{Sp}\big(\wedge^2\rho_{2,4}^{(t)}(\sigma_1)\big)=\{-t,1,-t\}=-t\{1,-t^{-1},1\}$. The second part follows from (2.2) and (2.3). $\square$

It would be nice to quantize and study the representations of B_n defined by (3.1)

$$\sigma_k\to\big[S^r\big(\exp(T_{nm})\circ\rho_n^{(t)}(\sigma_k)\big)\big]_q,\quad \sigma_k\to\big[\wedge^l\big(\exp(T_{nm})\circ\rho_n^{(t)}(\sigma_k)\big)\big]_q.\tag{3.4}$$

Declaration of competing interest

None declared.

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Data availability

No data was used for the research described in the article.

References

[1] S. Albeverio, A. Kosyak, q-Pascal's triangle and irreducible representations of the braid group B_3 in arbitrary dimension, arXiv:0803.2778v2 [math.QA(RT)].
[2] G.E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Addison–Wesley Publishing Company, Cambridge, Massachusetts, 1976.
[3] E. Artin, Theorie des Zöpfe, Abh. Math. Semin. Univ. Hamb. 4 (1926) 47–72.
[4] S. Bigelow, The Burrau representation of the braid group B_n is not faithful for n=5, Geom. Topol. 3 (1999) 397–404.
[5] S. Bigelow, Braid groups are linear, J. Am. Math. Soc. 14 (2) (2001) 471–486.
[6] S. Bigelow, The Lawrence-Krammer representation, arXiv:math.GT/0204057.
[7] S. Bigelow, Does the Jones polynomial detect the unknot?, J. Knot Theory Ramif. 11 (4) (2002) 493–505.
[8] J.S. Birman, Braids, Links and Mapping Class Groups, Ann. Math. Studies, vol. 82, Princeton Univ. Press, 1974.
[9] J.S. Birman, T.E. Brendel, Braids: a survey, in: W. Menasko, T. Thistlethwaite (Eds.), Handbook of Knot Theory, Elsevier, 2005.
[10] W. Burau, Über Zopfgruppen und gleichsinnig verdrillte Verkettungen, Abh. Math. Semin. Univ. Hamb. 11 (1936) 179–186.
[11] S.P. Humphries, Some linear representations of braid groups, J. Knot Theory Ramif. 9 (2000) 341–366.
[12] E. Formanek, W. Woo, I. Sysoeva, M. Vazirani, The irreducible complex representations of n string of degree $\leq n$, J. Algebra Appl. 2 (2003) 317–333.

[13] V.F.R. Jones, Hecke algebra representations of braid groups and link polynomials, Ann. Math. 126 (1987) 335–388.
[14] C. Kassel, Quantum Groups, Springer-Verlag, 1995.
[15] C. Kassel, V. Turaev, Braid Groups, Graduate Texts in Mathematics, vol. 247, Springer, New York, 2008 (340 p. + xii).
[16] A.V. Kosyak, Representations of the braid group B_n and the highest weight modules of $U(\mathfrak{sl}_{n-1})$ and $U_q(\mathfrak{sl}_{n-1})$, arXiv:0803.2785v2 [math.QA(RT)].
[17] D. Krammer, The braid group B4 is linear, Invent. Math. 142 (3) (2000) 451–486.
[18] D. Krammer, Braid groups are linear, Ann. Math. 155 (2) (2002) 131–156.
[19] M.J. Larsen, E.C. Rowell, An algebra-level version of a link-polynomial identity of Lickorish, Math. Proc. Camb. Philos. Soc. 144 (3) (2008) 623–638.
[20] R. Lawrence, Homological representations of the Hecke algebra, Commun. Math. Phys. 135 (1990) 141–191.
[21] W.B.R. Lickorish, Some link-polynomial relations, Math. Proc. Camb. Philos. Soc. 105 (1) (1989) 103–107.
[22] D. Long, M. Paton, The Burrau representation of the braid group B_n is not faithful for $n \geq 6$, Topology 32 (1993) 439–447.
[23] W. Magnus, A. Peluso, On a theorem of V.I. Arnold, Commun. Pure Appl. Math. 22 (1969) 683–692.
[24] J. Moody, The Burrau representation of the braid group B_n is not faithful for large n, Bull. Math. Soc. 25 (1991) 379–384.
[25] J.-P. Serre, Lie Algebras and Lie Groups, 1964 Lectures Given at Harvard University, Part of, Lect. Notes Math., 1500.
[26] R. Smeltzer, Linear representations of braid groups, A thesis for the degree of the master of sciences, McMaster University, July 2003, 65 p.
[27] A. Stoimenov, The density of Lawrence–Krammer and non-conjugate braid representation of links, arXiv:0809.0033 [math.GR].
[28] I. Tuba, Low-dimensional unitary representations of B_3, Proc. Am. Math. Soc. 129 (2001) 2597–2606.
[29] I. Tuba, H. Wenzl, Representations of the braid group B_3 and of $\mathrm{SL}(2, \mathbb{Z})$, Pac. J. Math. 197 (2) (2001) 491–510.
[30] B. Westbury, On the character varieties of the modular group, Nottingham, preprint, 1995.
[31] H. Weyl, The Classical Groups, Their Invariants and Representations, first edition, Princeton University Press, 1939. Second Edition with supplements, 1949.
[32] M. Zinno, On Krammer's representation of the braid group, Math. Ann. 321 (2001) 197–211.